Lecture 7. **Concept of the stability. Algebraic criteria of stability**

7.1 Stability. Stability according to A.M. Lyapunov

Notion *stability* of regulating system is related to its ability to return to equilibrium state after all external forces which lead to unstable state have vanished. So, *unstable system* does not return to equilibrium state out of which it was brought. A vivid example of stability of equilibrium is a ball lying in a cavity (fig. 3.1a, 3.1b, 3.1c).

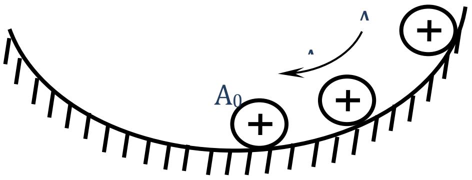

Fig. 3.1a. Stable equilibrium

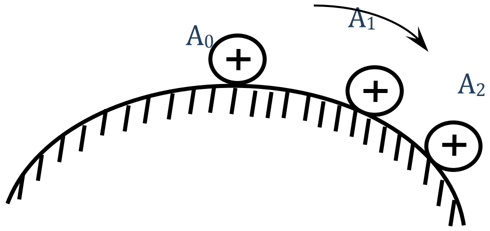

Fig. 3.1b. Unstable equilibrium

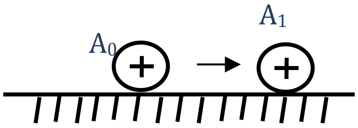

Fig. 3.1c. Neutral (indifferent) equilibrium

Any deflection of the ball from its equilibrium state will tend to decrease, thus returning either to its exact initial position (no friction) or to some finite equilibrium region (in presence of friction forces). Such position of the ball is stable (fig. 3.1).

Let us introduce a notion of Undisturbed Equilibrium state, corresponding to point A_0 (all points in this paragraph are from fig. 3.1a – 3.1c), and Disturbed Equilibrium state, corresponding to point A_2. After all external forces are vanished the ball returns to point A_0 or A_1. Stability criterion in this case can be formulated as: if the system passes from disturbed equilibrium state to some finite region surrounding undisturbed state, then it is called a *stable system*.

Stability notion can be extended to cover dynamic systems too. In general, considering nonlinear systems, notions *stability in small*, *stability in large* and *global stability* are used. A system is *stable in small*, if only a stability region existence fact is established, but not its boundaries. In contrast, a system is called *stable in large*, if these boundaries are determined, i.e. the boundaries of initial deflection region at which the system returns to initial state are determined, and real initial deflections are within these boundaries. In a case when a system falls back to its equilibrium state at any initial deflections, it is called *generally stable.*

Stability notion is extended to more general case, when we consider a system movement in whole as its equilibrium, not only its undisturbed state.

Now, let a dynamic system state be defined as a set of independent coordinates $x_1(t),\ x_2(t),\ \ldots,\ x_n(t)$. The system movement is described as a set of rules $x_{10}(t),\ x_{20}(t),\ \ldots,\ x_{n0}(t)$. This predefined movement is called *Undisturbed*

Movement. Application of external disturbances will cause deflection of real behavior from the predefined one:

$$x_1(t) \neq x_{10}(t),\ x_2(t) \neq x_{20}(t),\ \ldots,\ x_n(t) \neq x_{n0}(t).$$

The real behavior is called *Disturbed Movement.*

Definition: predefined undisturbed motion of a system is called *Stable Motion* if as a result of application of external forces (which are eliminated after that) the system disturbed motion after some period of time move to a region

$$\left\| x_i(t) - x_{i0}(t) \right\| < \xi_i \quad \forall \xi_i > 0,\ \xi_i = const;\ \left(i = \overline{1,n}\right).$$

Let us now consider stability even more deeply. For the first time strict definition of stability was given by Russian scientist A. M. Lyapunov in 1892.

A little bit mathematics: let a dynamic system be described by a set of nonlinear differential equations in Cauchy form:

$$\frac{dx_i}{dt} = F_i(x_1,\ x_2,\ \ldots,\ x_n);\ \forall i == 1,\ldots,n;\ \ x = \begin{vmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{vmatrix}\ ; \qquad (3.1)$$

$$\dot{x} = F(x).$$

Here $\dot{x} = \begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ldots \\ \dot{x}_n \end{vmatrix}; \quad F(x) = \begin{vmatrix} F_1 \\ F_n \end{vmatrix}$ are nonlinear vector functions of vector argument.

If initial values x_{i0} are given at $t = t_0$ the solution can be written as

$$x_i = x_i(x_{10},\ x_{20},\ \ldots,\ x_{n0})^T,\ \ i = \overline{1,n}\,.$$

Let the system steady state be described by coordinates $x_i^0 = \left(x_1^0,\ \ldots,\ x_n^0\right)^T,\ \ \left(i = \overline{1,n}\right)$. Define coordinates deflection $\Delta x_i = x_i - x_i^0\ \left(i = \overline{1,n}\right)$ characterizing the deflection of real behavior from undisturbed steady state behavior. Then we can rewrite equations (3.1) in terms of these deflections:

$$\frac{d\Delta x_i}{dt} = f_i(\Delta x_1,\ \Delta x_2,\ \ldots,\ \Delta x_n),\ \left(i = \overline{1,n}\right) \qquad (3.2)$$

where f_i are some nonlinear functions.

Equations (3.2) are called *disturbed motion equations***.** Initial values of deflections Δx_{i_0}, $\left(i=\overline{1,n}\right)$ are called *disturbances*. The solution of (3.2) at particular $\Delta x_i = \Delta x_i\left(\Delta x_{10},\ \Delta x_{20},...,\ \Delta x_{n0}, t\right)$ is a *disturbed motion***.**

A.M. Lyapunov in his works gave the following definitions of stability.

The first definition: Undisturbed motion $\Delta x_i = 0$ is called *Stable according to Lyapunov* with respect to variables x_i if at any given positive infinitesimal $\xi > 0$ there exist positive $\gamma > 0$ such that for all Δx_{i0} if

$$\left\|\Delta x_{i0}\right\| < \gamma_i \tag{3.3}$$

then disturbed motion (3.2) for $t > t_0$ satisfies

$$\left\|\Delta x_i\right\| < \xi_i\,,\ \ i=\overline{1,n} \tag{3.4}$$

Here norm is: $\left\|\Delta x_i\right\| = \sqrt{\sum_{i=1}^{n} \Delta x_i^2}$.

The second definition: Undisturbed motion is called *Asymptotically Stable according to Lyapunov* if additional condition

$$\lim_{t\to\infty} \Delta x_i(t) = 0 \quad \left(i=\overline{1,n}\right) \tag{3.5}$$

is satisfied.

The third definition: Undisturbed motion is called *Unstable according to Lyapunov* if there exists moment of time $t = t_1 > t_0$ at which condition (3.4) is not satisfied, i.e.

$$\left\|\Delta x_i\right\| \geq \xi_i \quad \left(i=\overline{1,n}\right).$$

Geometrical interpretation

Figures 3.2a – 3.2d give graphical presentation of notion mentioned above.

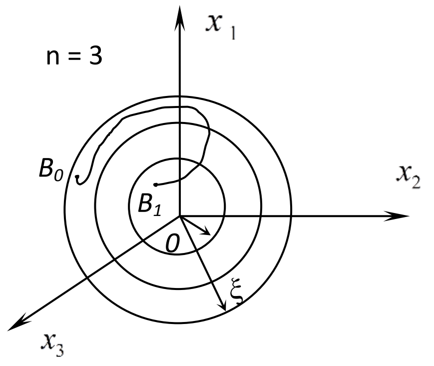

Fig. 3.2a. Stable motion

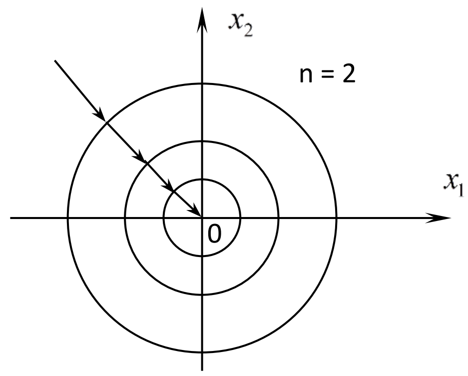

Fig. 3.2b. Asymptotically Stable motion

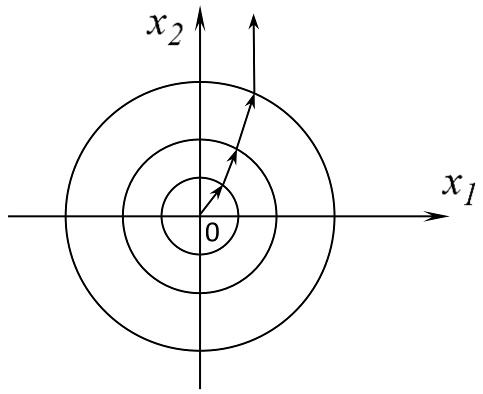

Fig. 3.2c. Unstable motion

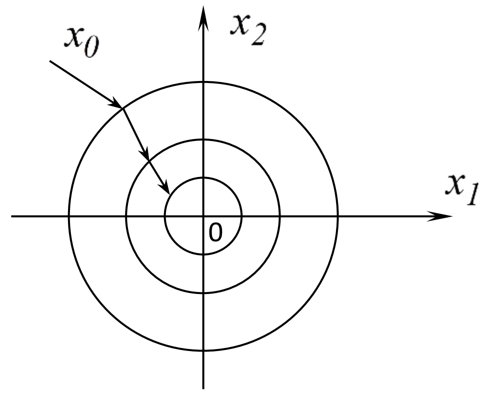

Fig. 3.2d. Stable motion

Physical meaning

Simply stated, linear system is stable if its response to any finite action is also finite, and is unstable if the response is infinite. Stability is essential for normal operation of dynamic control systems.

7.2 Algebraic criteria of stability

In this part we will deal with the question of linear (linearized, to be more precise) dynamic control systems stability.

Let a dynamic system be described by a set of constant coefficient differential equations in terms of input and output signals:

$$\begin{aligned} a_0 \frac{d^n x_{out}}{dt^n} + a_1 \frac{d^{n-1} x_{out}}{dt^{n-1}} + \ldots + a_n x_{out}(t) = \\ = b_0 \frac{d^m x_{in}}{dt^m} + b_1 \frac{d^{m-1} x_{in}}{dt^{m-1}} + \ldots + b_m x_{in}(t) \quad (m \le n) \end{aligned} \quad . \qquad (3.9)$$

Characteristic equation of (3.9) is of the form

$$Q_2(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_n = 0 \; . \qquad (3.10)$$

Roots s_1, s_2, ..., s_n define nature of transition process. Pay attention to the fact that letter *s* is used in characteristic equation as a complex number, not as a symbol of differentiation.

In general, roots of the characteristic equation are in the form: $s_i = \alpha_i \pm j\beta_i$ $(i = \overline{1,n})$; $j = \sqrt{-1}$. In fig. 3.4 you can see possible positions of roots in the complex root plane S, at

$s_1 = \alpha_i$; $s_2 = \alpha_2 + j\beta_2$; $s_3 = \alpha_2 - j\beta_2$; $s_4 = 0$; $s_5 = -\alpha_5$; $s_6 = -\alpha_6 + j\beta_6$; $s_7 = -\alpha_6 - j\beta_6$.

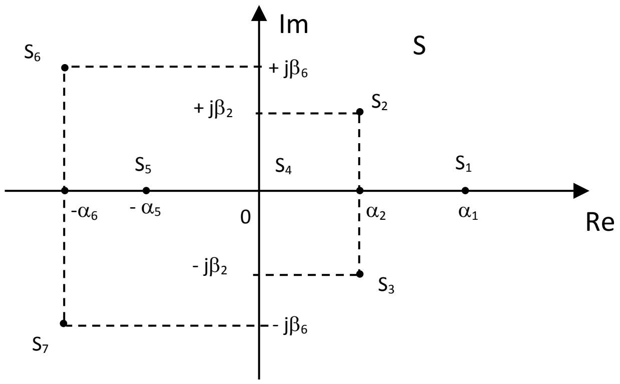

Fig. 3.4. Various positions of roots on complex plane

If all roots are different, then they are called *simple*, and if not — *divisible*. As a rule, *roots* with negative real component are called *left* because of their left position relatively to the imaginary axis (*"y"* axis). Obviously, *roots* with positive real component are called *right***.**

Stability condition of the linear system is formulated as the following: if all roots of the characteristic equation are left, then the system is asymptotically stable (this condition is necessary and sufficient).

Fortunately, if we have to determine, whether dynamic system is stable or not, there is no need to solve characteristic equation. There are some rules, called *stability criteria*, which can help us to determine stability of the system using characteristic equation coefficients.

Stability criteria can be divided into algebraic and frequency. The most popular algebraic stability criteria are the ones of Routh and Hurwitz. We emphasize, that positive coefficients $a_0 > 0$; $a_1 > 0$; … $a_n > 0$ of characteristic equation are necessary condition for system stability.

Let us consider characteristic equation of linearized dynamic system:

$$Q_2(s) = a_0 s^n + a_2 s^{n-1} + ... + a_n = 0.$$

Routh's stability criterion

British mathematician E. J. Routh proposed his algorithm in 1877, stated as the following: *for a system to be stable it is necessary and sufficient for all coefficients of the first column of Routh table to be positive, i.e.* $c_{v1} > 0, \quad v = \overline{1,n}$.

Hence, if there is at least one negative coefficient, then the whole system is unstable.

Coefficients of the first column of Routh table are defined in the following way:

$$c_{v1} = \frac{\Delta_{v-1}}{\Delta_{v-2}},$$

where $\Delta_v (v = 3,\ 4,\ ...,\ n)$ are diagonal determinants, obtained from Hurwitz matrix, which is of $(n \times n)$ dimension:

$$\Delta(n \times n) = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & a_9 & ... & 0 \\ a_0 & a_2 & a_4 & a_6 & a_8 & ... & 0 \\ 0 & a_1 & a_3 & a_5 & a_7 & ... & 0 \\ 0 & a_0 & a_2 & a_4 & a_6 & ... & 0 \\ 0 & 0 & a_1 & a_3 & a_5 & ... & 0 \\ 0 & 0 & a_0 & a_2 & a_4 & ... & a_n \end{vmatrix}$$

Hurwitz matrix is constructed in the following manner: the first row is filled with odd-indexed constant coefficients of characteristic equation; the second row is filled with even-indexed coefficients. The third and fourth rows are the first and second rows (correspondingly) shifted rightwards by one position, and so on.

Hurwitz's criterion of stability

In 1895 A. Hurwitz (German mathematician) proposed, that *for any closed-loop system to be stable it is necessary and sufficient, that having* $a_0 > 0$ *all minors of Hurwitz determinant must be strictly positive,* i.e.

$$\Delta_1 = a_1 > 0;\ \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = a_1 a_2 - a_0 a_3 > 0;\ \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0.$$

Thus, positive coefficients of the characteristic equation (of degree 1 or 2) are necessary and sufficient condition for dynamic system stability.

The last determinant, as you can see, encapsulates the whole matrix. But all elements of the last column, except the last one, are zeroes. Hence, the last Hurwitz' determinant is found using previous one in the following way:

$$\Delta_n = a_n \Delta_{n-1} > 0. \qquad (3.11)$$

Since in stable system next to last element is positive $\Delta_{n-1} > 0$, then $a_n > 0$.
In control theory three stability thresholds are accepted.

We can determine whether a system is on one of thresholds by making last determinant be equal to zero (taking into account that all other determinants are positive). From inequality (3.11) we can obtain two conditions: $a_n = 0,\ \Delta_{n-1} = 0$. *Condition* $a_n = 0$ *corresponds to the stability threshold of the first type — zero root or aperiodic stability threshold. Condition* $\Delta_{n-1} = 0$ *corresponds to the stability threshold of the second type — oscillating stability Threshold. Stability threshold of the third type is infinite root, i.e.* $S_k = \infty$.